# 나눗셈의 극한은 극한의 나눗셈이다.

(The limit of a quotient is the quotient of the limits)

# The limit of a quotient is the quotient of the limits

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Start End

**Theorem**

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**Theorem**

$\lim_{x \to a} f(x) = L$

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**Theorem**

$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M (\neq 0)$

# The limit of a quotient is the quotient of the limits

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**Theorem**

$\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M(\neq 0)$

$$\lim_{x \to a} \left\{ \frac{f(x)}{g(x)} \right\}$$

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## Theorem

$\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M (\neq 0)$

$$\lim_{x \to a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{L}{M}$$

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Proof.

## The limit of a quotient is the quotient of the limits

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### Proof.

Show that $\lim_{x \to a} \left\{ \frac{1}{g(x)} \right\} = \frac{1}{M}$

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Show that $\lim_{x \to a} \left\{ \frac{1}{g(x)} \right\} = \frac{1}{M}$

$\epsilon > 0$

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Show that $\lim_{x \to a} \left\{ \frac{1}{g(x)} \right\} = \frac{1}{M}$

$\epsilon > 0 \quad \left| \frac{1}{g(x)} - \frac{1}{M} \right|$

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$\exists\delta_1 > 0$

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$\delta$

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Start End

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## The limit of a quotient is the quotient of the limits

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$\lim_{x \to a} \left\{ \frac{f(x)}{g(x)} \right\}$

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Github:
https://min7014.github.io/math20240105001.html

Click or paste URL into the URL search bar, and you can see a picture moving.